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ON A NON-CORRECT PROBLEM FOR A BIHARMONIC EQUATION IN A SEMICIRCLE

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Abstract
This work investigates a conditionally correct problem for a biharmonic equation in a semicircular domain. The problem involves finding a function satisfying a biharmonic equation with mixed boundary conditions, including Dirichlet and Neumann-type constraints. It is demonstrated that the solution does not depend continuously on the input data, confirming the ill-posed nature of the problem. A stability estimate for the solution is derived under an a priori bound, and a family of regularizing operators is introduced to construct approximate solutions from noisy data. The effectiveness of the regularization method is analyzed, and an optimal parameter choice is discussed. An auxiliary problem is also formulated and reduced to a Fredholm integral equation of the first kind, which is addressed using Tikhonov regularization.

Keywords: Biharmonic equation, semicircle, ill-posed problem, conditional correctness, stability estimate, regularization method, Tikhonov regularization, Fredholm integral equation.

Introduction

Problems for higher-order partial differential equations, such as the biharmonic equation, arise in various fields of mathematical physics, including elasticity, plate theory, and fluid dynamics. While well-posed problems for such equations have been extensively studied, ill-posed problems—where solutions may not

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exist, be unique, or depend continuously on data—require specialized analytical and numerical approaches.

In this paper, we examine a non-standard boundary value problem for the biharmonic equation in a semicircular domain. The problem is characterized by a combination of boundary conditions on the radial and angular coordinates, including vanishing values of the solution and its Laplacian on the straight edges, as well as Dirichlet and Neumann conditions on the circular arc. Such formulations often emerge in inverse and hybrid problems, where part of the boundary data is overspecified or incomplete.

It is shown that the problem is conditionally correct: a continuous dependence of the solution on the data is absent in general, but stability can be restored under an a priori bound on the solution norm. We establish a quantitative stability estimate and propose a regularization strategy based on truncation of Fourier series. The accuracy of the approximate solution is estimated in terms of the noise level and the regularization parameter. Additionally, an auxiliary problem is introduced and transformed into a Fredholm integral equation of the first kind, which is treated via Tikhonov regularization.

The study extends earlier works on ill-posed problems for biharmonic equations in different geometries and contributes to the broader theory of conditional correctness and regularization for higher-order PDEs.

In this work, an approximate solution of one problem for a biharmonic equation in a semicircle is studied for conditional correctness.

1. Task. You want to find a function $U(\rho,\varphi)$ that meets the following conditions:

$$\Delta^2 U(\rho,\varphi) = 0 \text{ in } D = \left\{ (\rho,\varphi) : 0 < \rho < b,\ 0 \le \varphi \le \frac{\pi}{2} \right\} \quad (1)$$

$$U(\rho,0) = U\left(\rho,\frac{\pi}{2}\right) = 0,\ 0 \le \rho \le b, \quad (2)$$

$$\Delta U(\rho,0) = \Delta U(\rho,\frac{\pi}{2}) = 0,\ 0 < \rho < b, \quad (3)$$

$$U(a,\varphi) = 0,\ 0 \le \varphi \le \frac{\pi}{2}, \quad (4)$$

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$$\frac{\partial U(a,\varphi)}{\partial \rho} = f(\varphi), \quad 0 < \varphi < \frac{\pi}{2}, \qquad (5)$$

where 0< a < b, $f(\varphi)$ is a given function, $\Delta -$ a Laplace operator.

2. Let us show that in the problem there is no continuous dependence of the solution on the data. Indeed, the function

$$U_m(\rho,\varphi) = \varepsilon \frac{\rho^2 - a^2}{2a} \left(\frac{\rho}{a}\right)^m Sin\, m\varphi \qquad (6)$$

is the solution of problem (1)-(5) with $f(\varphi) = \varepsilon Sin 2m\varphi$.

It follows from (6) that for any constants $0 < \varepsilon < 1$, $c > 0$ and variables $\varphi \in (0, \frac{\pi}{2})$,, it $\rho \in (a,b)$ is possible to select such ε and m so that the inequalities are satisfied

$$\|\varepsilon\, Sin\, m\varphi\|_{L_2(0,\frac{\pi}{2})} \le \varepsilon; \qquad \|U_m(\rho,\varphi)\|_{L_2(0,\frac{\pi}{2})} > c.$$

3. The following theorem is valid, characterizing the stability of the solution of problem (1)-(5).

Theorem. If the function $U(\rho,\varphi)$ satisfies the relations:

$$\|U(\rho,\varphi)\|_{L_2(0,\frac{\pi}{2})} \le M, \qquad (7)$$

$$\left\|\frac{\partial U(a,\varphi)}{\partial \rho}\right\|_{L_2(0,\frac{\pi}{2})} \le \varepsilon, \qquad (8)$$

$$U(a,\varphi) = 0,\ 0 \le \varphi \le \frac{\pi}{2}, \qquad (9)$$

$$\Delta U(\rho,0) = U(\rho,0) = U(\rho,\frac{\pi}{2}) = \Delta U(\rho,\frac{\pi}{2}) = 0,\ \ 0 \le \rho \le b, \qquad (10)$$

then the inequality is fulfilled

$$\|U(\rho,\varphi)\|_{L_2(0,\frac{\pi}{2})} \le \frac{|\rho^2 - a^2|}{b^2 - a^2} \cdot M \left(\frac{\rho}{b}\right)^{2\lambda(\varepsilon)}, \qquad (11)$$

where $\lambda(\varepsilon)$ is the root of the equation

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$$\frac{b^2-a^2}{2a}\left(\frac{b}{a}\right)^{2\lambda}=\frac{M}{\varepsilon} \qquad (12)$$

Proof. The solution of problem (1)-(5) can be written in the form of:

$$U(\rho,\varphi)=\frac{\rho^2-a^2}{2a}\sum_{k=1}^{\infty}\left(\frac{\rho}{a}\right)^{2k} a_k \sin 2k\varphi \qquad (13)$$

From (7), (8), (13) it follows that

$$\|U(\rho,\pi)\|_{L_2(0,\frac{\pi}{2})}=\frac{4}{\pi}\int_0^{\frac{\pi}{2}}U^2(\rho,\varphi)\,d\varphi=\frac{\left(\rho^2-a^2\right)^2}{4a^2}\sum_{k=1}^{\infty}\left(\frac{\rho}{a}\right)^{4k} a_k^2 \qquad (14)$$

$$\sum_{k=1}^{\infty}\left(\frac{b}{a}\right)^{4k} a_k^2 \le M^2 \qquad (15)$$

$$\sum_{k=1}^{\infty} a_k^2 \le \varepsilon^2 \qquad (16)$$

The sum in the right-hand side (14) reaches a conditional maximum at $C_k=0$, $k\neq p,q$ and C_p , C_q satisfies one of the three ratios [1]:

$$\left.\begin{aligned} & a_p^2+a_q^2=\varepsilon^2 \\ & \left(\frac{b}{a}\right)^{4p} a_p^2+\left(\frac{b}{a}\right)^{4q} a_q^2=\frac{4a^2}{b^2-a^2}M^2 \end{aligned}\right\}, \qquad (17)$$

$$a_p=0, \qquad (18)$$

$$a_q=0, \qquad (19)$$

where p, q (p<q) are some numbers.

Let the ratio (17) take place. Then

$$a_p^2=\frac{\varepsilon^2\left(\frac{b}{a}\right)^{4q}-\frac{4a^2}{b^2-a^2}M^2}{\left(\frac{b}{a}\right)^{4q}-\left(\frac{b}{a}\right)^{4p}}\ge 0 \qquad (20)$$

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$$a_q^2 = \frac{\frac{4a^2}{b^2-a^2}M^2 - \varepsilon^2\left(\frac{b}{a}\right)^{4p}}{\left(\frac{b}{a}\right)^{4q} - \left(\frac{b}{a}\right)^{4p}} \geq 0 \qquad (21)$$

From (20),(21) it follows that

$$\frac{b^2-a^2}{2a}\left(\frac{b}{a}\right)^{2p} \leq \frac{M}{\varepsilon} \leq \frac{b^2-a^2}{2a}\left(\frac{b}{a}\right)^{2q} \qquad (22)$$

By virtue of (13), (19) - (21) we get

$$\|U(\rho,\varphi)\|_{L_2(0,\frac{\pi}{2})} \leq \frac{\left|\rho^2-a^2\right|}{b^2-a^2}\cdot M\left(\frac{\rho}{b}\right)^{2\lambda(\varepsilon)}, \qquad (23)$$

where $\lambda(\varepsilon)$ is the root of the equation (12)

Let there now be a relation (18). Then

$$\|U(\rho,\varphi)\|^2_{L_2(0,\frac{\pi}{2})} = \frac{\rho^2-a^2}{4a^2}\left(\frac{\rho}{a}\right)^{4q} a_q^2$$

and by virtue of (15), (16)

$$a_q^2 \leq \varepsilon^2$$

$$\frac{b^2-a^2}{4a^2}\left(\frac{b}{a}\right)^{4q} a_q^2 \leq M^2$$

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$$\|U(\rho,\varphi)\|_{L_2(0,\frac{\pi}{2})} \leq \frac{\rho^2-a^2}{b^2-a^2} M\left(\frac{\rho}{b}\right)^{2\lambda(\varepsilon)}, \qquad (24)$$

where $\lambda(\varepsilon)$ is the root of the equation (12).

In the case of ratio (19), inequality (24) is similar.

The statement of the theorem follows from (23) and (24).

4. Consider a family of linear operators B_n dependent on an integer parameter, defined as follows:

$$B_n f(\varphi) = \frac{\rho^2-a^2}{2a}\sum_{k=1}^{n} a_k\left(\frac{\rho}{a}\right)^{4k} \sin 2k\varphi; \qquad (25)$$

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here a_k are the Fourier coefficients of the function $f(x)$. The family of operators B_n will be regular, if $f(\varphi)$ $U(\rho,\varphi)$ the solution is also considered as elements of Hilbert spaces $L_2(0,\frac{\pi}{2})$ [2]. Now we get an estimate of the efficiency of applying this family to the solution of the problem of constructing an approximate solution from approximate data. Suppose that the problem (1)-(5) is conditionally correct and the set of correctness is determined by the inequality (6).

Let it $f(\varphi)$ be known with precision δ, i.e. the element $f_\delta(\varphi)$:

$$\|f(\varphi)-f_\delta(\varphi)\|_{L_2(0,\frac{\pi}{2})} \le \delta \qquad (26)$$

Let us take as an approximate solution of problem (1)-(4) the function:

$$U_{n\delta}(\rho,\varphi)=B_n f_\delta(\varphi)=\frac{\rho^2-a^2}{2a}\sum_{k=1}^{n} a_k \left(\frac{\rho}{a}\right)^{4k} sin2k\varphi\,, \qquad (27)$$

Where is

$$a_k=\frac{4}{\pi}\int_0^{\frac{\pi}{2}} f_\delta(\varphi) sin2k\varphi d\varphi$$

The exact solution of problem (1)-(4) in the set of correctness (5) has the form:

$$U(\rho,\varphi)=\frac{\rho^2-a^2}{2a}\sum_{k=1}^{\infty} a_k \left(\frac{\rho}{a}\right)^{4k} sin2k\varphi\,; \qquad (28)$$

In here

$$a_k=\frac{4}{\pi}\int_0^{\frac{\pi}{2}} f_\delta(\varphi) sin2k\varphi d\varphi \qquad (29)$$

Let's estimate the difference between $U_{n\delta}(\rho,\varphi)$ and $U(\rho,\varphi)$:

$$\| U(\rho,\varphi)-U_{n\delta}(\rho,\varphi)\|_{L_2\left(0,\frac{\pi}{2}\right)}=$$

$$=\|U(\rho,\varphi)-U_n(\rho,\varphi)+U_n(\rho,\varphi)-U_{n\delta}(\rho,\varphi)\|_{L_2\left(0,\frac{\pi}{2}\right)} \le$$

$$\le \|U_n(\rho,\varphi)-U_{n\delta}(\rho,\varphi)\|_{L_2(0,\frac{\pi}{2})}+\|U(\rho,\varphi)-U_n(\rho,\varphi)\|_{L_2(0,\frac{\pi}{2})}=$$

$$=\| B_n f(\varphi)-B_n f_\delta(\varphi)\|_{L_2\left(0,\frac{\pi}{2}\right)}+\| B_n f(\varphi)-U(\rho,\varphi)\|_{L_2\left(0,\frac{\pi}{2}\right)}=$$

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$$=\| B_n[f_\delta(\varphi)-f(\varphi)]\|_{L_2\left(0,\frac{\pi}{2}\right)}+\| B_n[f(\varphi)-U(\rho,\varphi)]\|_{L_2\left(0,\frac{\pi}{2}\right)}\le$$
$$\le\delta\| B_n\|_{L_2\left(0,\frac{\pi}{2}\right)}+\| B_n f(\varphi)-U(\rho,\varphi)\|_{L_2\left(0,\frac{\pi}{2}\right)} \quad (30)$$

From (28) (29) it follows that

$$\|B_n\|_{L_{2(0,\frac{\pi}{2})}}=\frac{\rho^2-a^2}{2a}\left(\frac{\rho}{a}\right)^{2n}, \quad (31)$$

$$\|B_n f(\varphi)-U(\rho,\varphi)\|^2_{L_{2(0,\frac{\pi}{2})}}=\frac{\left(\rho^2-a^2\right)^2}{4a^2}\sum_{k=n+1}^{\infty}\left(\frac{\rho}{a}\right)^{4k}a_k^2 \quad (32)$$

Amount on the right side (32) provided (15)

$$\frac{b^2-a^2}{4a^2}\sum_{k=1}^{\infty}\left(\frac{b}{a}\right)^{4k}a_k^2\le M^2$$

reaches the maximum value when the coefficients a_k are equal to:

$$a_k=0,\quad k\neq n+1;\quad a_{k+1}=\frac{2aM}{b^2-a^2}\left(\frac{a}{b}\right)^{2(k+1)}$$

and, therefore,

$$\|B_n f(\varphi)-U(\rho,\varphi)\|_{L_{2(0,\frac{\pi}{2})}}\le\frac{\rho^2-a^2}{b^2-a^2}M\left(\frac{\rho}{b}\right)^{2(n+1)} \quad (33)$$

Consequently,

$$\left\|U(\rho,\varphi)-U_{n\delta}(\rho,\varphi)\right\|_{L_{2\left(0,\frac{\pi}{2}\right)}}\le\omega(M,n,\delta) \quad (34)$$

Where is

$$\omega(M,n,\delta)=\left(\rho^2-a^2\right)\left[\left(\frac{\rho}{a}\right)^{2n}\frac{\delta}{2a}+\left(\frac{\rho}{b}\right)^{2(n+1)}\frac{M}{b^2-a^2}\right]$$

Note that the effectiveness of regularization depends on the choice of the regularization parameter n, which can be determined from the equation

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$$\frac{b^2\left(a^2-b^2\right)}{2a\rho^2}\left(\frac{b}{a}\right)^{2n}=\frac{M}{\delta}.$$

With a fixed accuracy δ approximated to a given value of the parameter n at which it is achieved, $\inf \omega(M,n,\delta)$ it will be optimal in the sense of estimation (34).

5. Let the constant number M, which participates in the inequality (6), which determines the set of correctness of the problem (1)-(5), be unknown.

Let's consider the auxiliary task:

$$\Delta^2 U(\rho,\varphi)=0 \text{ in } D=\left\{(\rho,\varphi):0<\rho<b,\ 0\le\varphi\le\frac{\pi}{2}\right\} \quad (35)$$

$$\Delta U(b,\varphi)=g(\varphi),\ 0\le\varphi\le\frac{\pi}{2}, \quad (36)$$

$$U(b,\varphi)=0,\ 0\le\varphi\le\frac{\pi}{2}, \quad (37)$$

$$U(\rho,0)=U(\rho,\frac{\pi}{2})=0,\ 0\le\rho\le b, \quad (38)$$

$$\Delta U(\rho,0)=\Delta U(\rho,\frac{\pi}{2})=0,\ 0\le\rho\le b, \quad (39)$$

Problem (35)-(39) is correctly set and the solution to this problem is as follows:

$$U(\rho,\varphi)=\frac{\rho^2-a^2}{2a}\sum_{k=1}^{\infty}a_k\left(\frac{\rho}{a}\right)^{4k} sin2k\varphi; \quad (40)$$

In here

$$a_k=\frac{4}{\pi}\int_0^{\frac{\pi}{2}} f_\delta(\varphi)s\, sin2k\varphi\, d\varphi \quad (41)$$

The solution of the ill-posed problem (1)-(4) will be sought in the form of series (40), where $g(\varphi)$ it is considered as an unknown function. From condition (2) taking into account (41) we obtain the integral Fredholm equation of the first kind with respect to the function $g(\varphi)$:

$$\int_0^{\pi} K(\varphi,s)\,g(s)\,ds=f(\varphi), \quad (42)$$

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Where is

$$K(\varphi,s)=\frac{(a^2-b^2)}{2\pi}\sum_{k=1}^{\infty}\left(\frac{a}{b}\right)^k\frac{sink\varphi sinks}{k+1} \qquad (43)$$

An approximate solution (36) is constructed by the method of regularization by A.N. Tikhonov [3].

It should be noted that in the case when region D is the upper half-band, problem (1)-(5) is studied in [4].

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